Guideline of the answers
Mid Semester Test November 2010
Course : Matrix Algebra

1. (a) given $A = \begin{pmatrix} 2 & -3 & 0 \\ 4 & -1 & 3 \\ -1 & 0 & 1 \end{pmatrix}$

$A + B^T = (A - B)^T$
$A + B^T = A^T - B^T$

$$2B^T = A^T - A = \begin{pmatrix} 2 & 4 & -1 \\ -3 & -1 & 0 \\ 0 & 3 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -3 & 0 \\ 4 & -1 & 3 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 7 & -1 \\ -7 & 0 & -3 \\ 1 & 3 & 0 \end{pmatrix}$$

$$B^T = \begin{pmatrix} 0 & \frac{7}{2} & -\frac{1}{2} \\ -\frac{7}{2} & 0 & -\frac{3}{2} \\ \frac{1}{2} & \frac{3}{2} & 0 \end{pmatrix} \rightarrow B = \begin{pmatrix} 0 & -\frac{7}{2} & \frac{1}{2} \\ \frac{7}{2} & 0 & \frac{3}{2} \\ -\frac{1}{2} & -\frac{3}{2} & 0 \end{pmatrix}$$

(b). $\begin{pmatrix} x & y \\ w & s \end{pmatrix} = 2\begin{pmatrix} x & 3 \\ 2 & x+w \end{pmatrix} + \begin{pmatrix} 2+y & x+9 \\ w+s & y \end{pmatrix}$

$x = 2x + y + 2$ (1)
$y = 6 + x + 9$ (2)
$w = 4 + w + s$ (3)
$s = 2x + 2w + y$ (4)

from the equations, $x = -\dfrac{17}{2}$, $y = \dfrac{13}{2}$, $s = -4$, and $w = \dfrac{13}{4}$.

2. Suppose $A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ has an inverse matrix such that $A^{-1} = A$

$$AA^{-1} = AA = I \rightarrow \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a^2+bc & ab \\ ac & bc \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$a^2 + bc = 1$ (1)

$ab = 0$ (2)

$ac = 0$ (3)

$bc = 1$ (4)

from (4), it is clear that b and c are not zero, because $bc = 1 \rightarrow b = \frac{1}{c}$. Consequently, from (2) and (3) it's mean $a = 0$.

So, the matrix $\boldsymbol{A} = \begin{pmatrix} 0 & 1/c \\ c & 0 \end{pmatrix}$. For example : $\boldsymbol{A} = \begin{pmatrix} 0 & 1/4 \\ 4 & 0 \end{pmatrix}$, $\boldsymbol{A} = \begin{pmatrix} 0 & -3 \\ -1/3 & 0 \end{pmatrix}$, etc.

3. Given $\boldsymbol{A} = \begin{pmatrix} x & y \\ z & s \end{pmatrix}$

   $\boldsymbol{A}$ is orthogonal matrix $\rightarrow \boldsymbol{A}\,\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{I}$

   $\begin{pmatrix} x & y \\ z & s \end{pmatrix}\begin{pmatrix} x & z \\ y & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so the requirement for x, y, s, and s are :

   $x^2 + y^2 = 1$ (1)
   $xz + ys = 0$ (2)
   $z^2 + s^2 = 1$ (3)

   for example, from equation (1) if $x = \frac{1}{2}$, then $y = \pm\, \frac{1}{2}\sqrt{3}$; and from equation (3) if $s = \frac{1}{2}$, then $z = \pm\, \frac{1}{2}\sqrt{3}$; so based on equation (2), it must be taken (for example) $x = \frac{1}{2}$, $y = -\,\frac{1}{2}\sqrt{3}$, $s = \frac{1}{2}$, and $z = \frac{1}{2}\sqrt{3}$. thus the matrix $\boldsymbol{A} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$, or you can take the other combination which are satisfying the equation (1), (2), and (3) above.

4. Suppose $\boldsymbol{A} = \begin{pmatrix} -1 & x \\ y & 1 \end{pmatrix} \rightarrow \boldsymbol{A}\,\boldsymbol{A} = \boldsymbol{I}$

   $\begin{pmatrix} -1 & x \\ y & 1 \end{pmatrix}\begin{pmatrix} -1 & x \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

   $1 + xy = 1$
   $xy + 1 = 1 \rightarrow xy = 0$

   from that equation, $xy = 0$, it is mean that $x = 0$ or $y = 0$. Thus matrix $\boldsymbol{A} = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ -7 & 1 \end{pmatrix}$, or $\begin{pmatrix} -1 & 0 \\ 5 & 1 \end{pmatrix}$, etc.

5. $|K| = \begin{vmatrix} 1 & -3 & 4 & 2 \\ -4 & 7 & -2 & 5 \\ 0 & 6 & -3 & 4 \\ -2 & 5 & 2 & 3 \end{vmatrix}$ → applying B2 + 4B1, B4+2B1, B3 + $\frac{6}{5}$ B2, B4 – $\frac{1}{5}$ B2, then

B4 – $\frac{36}{69}$ B3 respectively which are obtained $|K| = \begin{vmatrix} 1 & -3 & 4 & 2 \\ 0 & -5 & 14 & 13 \\ 0 & 0 & \frac{69}{5} & \frac{98}{5} \\ 0 & 0 & 0 & -\frac{670}{115} \end{vmatrix} = 402.$

6. $\begin{vmatrix} -1 & 1 & y+1 \\ 0 & y & 2 \\ 1 & 1 & 0 \end{vmatrix} = -2$ → $y^2 + y - 6 = 0$ → $y = -3$ or $y = 2$.

Good Luck