Guide lines of the answer of FINAL TEST JANUARY 2012

Course : Matrix Algebra

Department/Semester : Mathematics Education / I

1. $|\boldsymbol{A}| = \begin{vmatrix} -3 & -1 & x & -2 \\ 1 & 1 & 2 & 7 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -3 & 2 \end{vmatrix} = -84$

$$= -3\begin{vmatrix} 1 & 2 & 7 \\ -1 & 1 & 0 \\ 1 & -3 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 & 7 \\ 2 & 1 & 0 \\ 0 & -3 & 2 \end{vmatrix} + x\begin{vmatrix} 1 & 1 & 7 \\ 2 & -1 & 0 \\ 0 & 1 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 0 & 1 & -3 \end{vmatrix} = -84$$

$= -3\,(20) + 1\,(-48) + x\,(8) + 2\,(12) = -84$

$8x = 0 \rightarrow x = 0.$

2. the system of linear equation : $\begin{cases} 4x_1 - 3x_2 - x_3 = 5 \\ -2x_1 + 2x_2 + x_3 = -1 \\ 2x_1 + 2x_2 - 4x_3 = 6 \end{cases}$, Using inverse of matrices:

$\boldsymbol{A} = \begin{pmatrix} 4 & -3 & -1 \\ -2 & 2 & 1 \\ 2 & 2 & -4 \end{pmatrix}$, $\boldsymbol{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, and $\boldsymbol{G} = \begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix}$

$\boldsymbol{A}^{-1} = \dfrac{1}{-14}\begin{pmatrix} -10 & -14 & -1 \\ -6 & -14 & -2 \\ -8 & -14 & 2 \end{pmatrix}$

$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{G} = \dfrac{1}{-14}\begin{pmatrix} -10 & -14 & -1 \\ -6 & -14 & -2 \\ -8 & -14 & 2 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$

Thus, $x_1 = 3$, $x_2 = 2$, and $x_3 = 1$.

3. $\begin{cases} 2x+4y+2z-s+7w=0 \\ x+2y+2z+s+2w=0 \\ -x-2y-z+s-3w=0 \end{cases}$, in a matrix is $\left(\begin{array}{cccc|c} 2 & 4 & 2 & -1 & 7 \\ 1 & 2 & 2 & 1 & 2 \\ -1 & -2 & -1 & 1 & -3 \end{array}\right)$, and echelon

form of the matrix is $\left(\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$, rank = 3, number of free variables = $n - r$ = 5 – 3 = 2, i.e y and w. Suppose $y = \alpha$ and $w = \beta$, then $s = -\beta$, $z = 3\beta$, and $x = -2\alpha - 7\beta$. Thus, the general solution is $\{(-2\alpha - 7\beta, \alpha, 3\beta, -\beta, \beta)\}$ where α and β are real number. Suppose $\alpha = 1$ and $\beta = 1$, one of the nontrivial solutions is $\{(-9, 1, 3, -1, 1)\}$.

4. $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & 2 & 6 \\ 1 & 2 & 4 & 4 \\ 3 & 7 & 10 & 14 \end{pmatrix}$. Applying elementary row operations $H_{21(-1)}$, $H_{31(-3)}$, and $H_{32(-2)}$ respectively to matrix $\boldsymbol{A}$ become $\boldsymbol{U} = \begin{pmatrix} 1 & 3 & 2 & 6 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So, $\boldsymbol{L} = \boldsymbol{E}_{21(1)}\, \boldsymbol{E}_{31(3)}\, \boldsymbol{E}_{32(2)}$ =

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

5. the system of linear equations $\begin{cases} 2x_1 - 3x_2 + x_3 = 1 \\ -x_1 + 2x_2 - x_3 = 0 \\ 3x_1 + x_2 - 2x_3 = 9 \end{cases}$, Using the Cramer rule :

$|\boldsymbol{A}| = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 2 & -1 \\ 3 & 1 & -2 \end{vmatrix} = 2$, $|\boldsymbol{A}1| = \begin{vmatrix} 1 & -3 & 1 \\ 0 & 2 & -1 \\ 9 & 1 & -2 \end{vmatrix} = 6$, $|\boldsymbol{A}2| = \begin{vmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & 9 & -2 \end{vmatrix} = 4$, $|\boldsymbol{A}3| = \begin{vmatrix} 2 & -3 & 1 \\ -1 & 2 & 0 \\ 3 & 1 & 9 \end{vmatrix} = 2$. Thus, $x_1 = \dfrac{|A1|}{|A|} = 3$, $x_2 = \dfrac{|A2|}{|A|} = 2$, and $x_3 = \dfrac{|A3|}{|A|} = 1$.

6. $(\boldsymbol{B}|\boldsymbol{I_3}) = \left(\begin{array}{ccc|ccc} 3 & -2 & -1 & 1 & 0 & 0 \\ 2 & -1 & -3 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$, applying elementary row operations H_{13}, $H_{21(-2)}$, $H_{31(-3)}$, $H_{32(-1)}$, and $H_{12(1)}$ respectively, become $\left(\begin{array}{cccc|cc} 1 & 0 & -5 & 0 & 1 & -1 \\ 0 & 1 & -7 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{array}\right) = (\boldsymbol{U}|\boldsymbol{P})$.

Now, $\left(\dfrac{U}{I_3}\right) = \left(\begin{array}{ccc} 1 & 0 & -5 \\ 0 & 1 & -7 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$, applying elementary column operations $K_{31(5)}$ and $K_{32(7)}$ respectively, become $\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \hline 1 & 0 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{array}\right) = \left(\dfrac{N}{Q}\right)$. Thus, $\boldsymbol{P} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -2 \\ 1 & -1 & -1 \end{pmatrix}$, $\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}$, and $\boldsymbol{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}$.

7. $\begin{cases} x+3y+z+2w=3 \\ 2x+y+2z+3w=10 \\ x-2y+z+w=7 \\ x+y+3z+w=8 \end{cases}$, in a matrix form is $\begin{pmatrix} 1 & 3 & 1 & 2 & 3 \\ 2 & 1 & 2 & 3 & 10 \\ 1 & -2 & 1 & 1 & 7 \\ 1 & 1 & 3 & 1 & 8 \end{pmatrix}$, and echelon form of the matrix is $\begin{pmatrix} 1 & 3 & 1 & 2 & 3 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{3}{10} & \frac{17}{10} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, rank = 3, number of free variables = $n - r$ = $4 - 3 = 1$, i.e w. Suppose $w = \alpha$, then $z = \dfrac{17+3\alpha}{10}$, $y = \dfrac{-8-2\alpha}{10}$, and $x = \dfrac{37-17\alpha}{10}$.

Thus, general solution is $\{(\frac{37-17\alpha}{10}, \frac{-8-2\alpha}{10}, \frac{17+3\alpha}{10}, \alpha)\}$, where α is real number. Suppose α = 1, one of the special solutions is {(2, -1, 2, 1)}.

8. If $\boldsymbol{A}^T$ and $\boldsymbol{A}^{-1}$ are transpose and inverse of $\boldsymbol{A}$ respectively, prove that $(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^{-1}$!

Proof:

According to properties of determinant $|\boldsymbol{A}^T| = |\boldsymbol{A}| \neq 0$, so inverse of A^T, i.e $(\boldsymbol{A}^T)^{-1}$ is exist; $(\boldsymbol{A}^T)^{-1}\ \boldsymbol{A}^T = \boldsymbol{A}^T\ (\boldsymbol{A}^T)^{-1} = \boldsymbol{I}$

Meanwhile, according to properties of transpose:

$(\mathbf{A}\ \boldsymbol{A}^{-1})^T = (\boldsymbol{A}^{-1})^T\ \boldsymbol{A}^T$

$\boldsymbol{I}^T = (\boldsymbol{A}^{-1})^T\ \boldsymbol{A}^T$

$(\boldsymbol{A}^{-1})^T\ \boldsymbol{A}^T = \boldsymbol{I}$, this mean that $(\boldsymbol{A}^{-1})^T$ is inverse of $\boldsymbol{A}^T$. Because inverse is unique, so $(\boldsymbol{A}^{-1})^T = (\boldsymbol{A}^T)^{-1}$.

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